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Relativistic Particle Mechanics

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Utilizing some ideas originating from R. Penrose's twistor theory we develop a (quasi) classical, relativistic but *not* field-theoretical formalism for the description of particle interactions. A simple example based on our ideas is explicitly solved. The physical interpretation of some very well-known relativistic notions is also given.

1. INTRODUCTION

Nonrelativistic quantum physics has its formal roots in the symplectic (Hamiltonian) structure of the phase space $F = \{(p_1, p_2, \ldots, p_N; q_1, q_2, \ldots, q_N)\}$. p_k and q_k denote a component of the generalized "linear momentum" and a coordinate of the generalized "position vector," respectively. The two variables (p_k, q_k), describing dynamics of the kth participating particle, constitute a pair of so-called canonically conjugate observables. Specifying a Hamiltonian $H = H(q_1, \ldots, q_N; p_1, \ldots, p_N)$ for, e.g., a closed system of particles we get equations of motion:

$$\frac{dp_k}{dt} = \{H, p_k\}, \quad \frac{dq_k}{dt} = \{H, q_k\}$$

where t is the time parameter and brackets denote Poisson brackets. Some years ago it was shown by Penrose (1968) that a similar Hamiltonian structure may be associated with the "space" of relativistic particles in interaction. This symplectic (canonical) structure is, however, apparently hidden from the direct insight (the reason for this seems to be of great significance)[1] and appears on the level of massless "square root" (i.e., twistor) description of particles. The canonically conjugate variables are twistors and twistor complex conjugates themselves. Replacing Poisson brackets by commutators the Poincaré algebra arises as a quadratic closed subalgebra lying in the (conformally broken) enveloping algebra of twistor operators.

These facts justify the approach presented in this note which is organized as follows. In the next section we introduce some relativistic concepts and definitions. In the third and fourth sections we interpret their physical content. In the fifth section we convert our quantities into relativistic quantum operators (Poincaré algebra). Some peculiar relativistic commutation relations between physical observables are derived. In the sixth section, using commutation relations of the Poincaré algebra, we generate interaction between operators corresponding to two different relativistic (sub)particles coupled together to form a closed spinning system. Specifying the Hamiltonian operator, performing all commutations and going to the relativistic classical limit we obtain a set of equations describing the motion of our (now classical) (sub)particles. A simple example is solved and some suggestions for the interpretation of solutions are made. Finally, we make a few comments on the present status of Penrose's twistor theory and also on possible further development of the ideas presented in this note. We wish also to emphasize that our exposition is in a certain sense self-contained and does not require any knowledge of twistor theory.

2. CONCEPTS AND DEFINITIONS

Let us imagine an inertial space origin O endowed with an intrinsic time parameter τ. Denote events at O by (O, τ). Next, assume that relative to O a remote relativistic massive system (particle) S is ultimately characterized by ten numbers evaluated with respect to the time coordinate τ and an arbitrary orthonormal space coordinate system also at O. The quantities represented by these numbers are: $E(\tau)$—energy of S, $\bar{p}(\tau)$—linear momentum of S and additionally two vectors $\bar{J}(\tau)$, $\bar{K}(\tau)$ associated with S, the precise physical meaning of which will be discussed shortly. In

[1]Twistor theory "predicts" the existence of correct internal particle symmetries and shows that they are a consequence of the conformal symmetry breaking, reducing conformal symmetry to (massive) Poincaré symmetry.

order to simplify computations we introduce four-dimensional tensor notation and put[2]

$$M^{\alpha 0} = M_{0\alpha} = -M_{\alpha 0} = -M^{0\alpha} = K_\alpha, \qquad \bar{K} = (K_1, K_2, K_3)$$

$$M^{\alpha\beta} = M_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma} J_\gamma, \qquad \bar{J} = (J_1, J_2, J_3)$$

$$-P^\alpha = P_\alpha = p_\alpha, \qquad P^0 = P_0 = E/c, \qquad \bar{p} = (p_1, p_2, p_3)$$

$$\varepsilon_{\alpha\beta\gamma} = \varepsilon_{[\alpha\beta\gamma]}, \qquad \varepsilon_{321} = 1 \qquad P^i P_i = m^2 c^2$$

The ten numbers are now given by $M^{ik} = -M^{ki}$ and P^i, Latin indices running over 0, 1, 2, 3 and Greek indices referring to space coordinates.

With respect to boosts and rotations of the orthonormal space coordinate system at (O, τ), M^{ik} and P^i transform as a second-rank 4-tensor and a 4-vector, respectively.

3. THE FREE-PARTICLE CASE

If the system S is closed (free particle) the quantities mentioned above become time independent.

Let us define a set of physically equivalent translated origins $\{O'\}$ (all parameterized by τ) and independently a set of physically equivalent "translated" intrinsic time scales $\{\tau + A^0/c: A^0/c\varepsilon R^1\}$ at $\{O'\}$. If we perform one such a general translation then the ten time-independent numbers M^{ik}, P^i mix with each other and transform according to the very well-known rule

$$M'^{ik} = M^{ik} - 2A^{[i}P^{k]}, \qquad P'^i = P^i \tag{3.1}$$

where $(A^1, A^2, A^3) = \bar{A} = \mathbf{OO'}$.

M'^{ik}, P'^i, being still time independent, once again characterize S but now with respect to the new (translated) space coordinate system at the point O' endowed with the new ("translated") intrinsic time parameter $\tau' = \tau + A^O/c$.

If at O we wish to assign space-time coordinates to the closed system S we may proceed as in the work of Penrose and MacCallum (1973). The relativistic centre of the total energy of S is thus defined as the locus of all

[2]Summation convention is assumed throughout this paper. Square brackets around sub- and superscripts denote antisymmetrization.

translated events $(\tilde{O}, X^O/c)$ such that $\tilde{P}_i\tilde{M}^{ik}=O$, i.e.,

$$\tilde{P}_i\tilde{M}^{ik}=P_iM^{ik}-P_iX^iP^k+m^2c^2X^k=0 \tag{3.2}$$

were $\bar{X}=\mathbf{O}\tilde{\mathbf{O}}=(X^1, X^2, X^3)$. Solving for X^i, with λ being an arbitrary parameter, we get

$$X^i=(1/m^2c^2)M^{ik}P_k+\lambda P^i \tag{3.3}$$

Splitting (3.3) into space and time coordinates we obtain

$$ct=\bar{K}\cdot\bar{p}/m^2c^2+\lambda E/c$$

$$\bar{r}=E\bar{K}/m^2c^3+\bar{J}\times\bar{p}/m^2c^2-\lambda\bar{p} \tag{3.4}$$

or equivalently with $\gamma=1/(1-v^2/c^2)^{1/2}$ and $\bar{v}=c^2\bar{p}/E$:

$$\gamma ct=-\gamma\bar{v}\cdot\bar{r}/c+c\lambda m$$

$$(1/\gamma)\bar{r}=\bar{K}/mc+\bar{J}\times\bar{v}/mc^2-\lambda m\bar{v} \tag{3.4'}$$

If $v/c\ll 1$ we may neglect all terms of order v^2/c^2 or higher and arrive at

$$ct\simeq-\bar{v}\cdot\bar{r}/c+c\lambda m$$

$$\bar{r}\simeq\bar{K}/mc-\lambda m\bar{v} \tag{3.5}$$

The formula above justifies the following identification:

$$\lambda m=\tau \tag{3.6}$$

Written out again the equations in (3.4) read

$$ct=-\bar{K}\cdot\bar{p}/m^2c^2+c\tau E/mc^2=-\bar{v}\cdot\bar{r}/c+c\tau(1/\gamma)$$

$$\bar{r}=E\bar{K}/m^2c^3+\bar{J}\times\bar{p}/m^2c^2-\tau\bar{p}/m=\gamma\bar{K}/mc+\gamma\bar{J}\times\bar{v}/mc^2-\tau\gamma\bar{v}$$

$$\tag{3.4''}$$

As is seen from (3.5) the vector $\bar{K}/mc$ forms an (approximate) separation vector between S and O at the time $\tau=0$. Alternatively we could say that $\bar{K}/mc$ constitutes a separation vector between S and an observer in rest relative S at (O, τ).

In order to generalize the above notion of separation vector to the relativistic case we perform a boost $c^2\bar{p}/E$ at (O,τ) and from (3.4″) obtain

$$
\begin{aligned}
t' &= \tau \\
\bar{r}' &= \bar{K}'/mc
\end{aligned}
\tag{3.7}
$$

$\bar{K}'/mc$ defines the searched-for separation vector, where $\bar{K}'$ is given by

$$\bar{K}' = \gamma\bar{K} + (\bar{J}\times\bar{p})/mc - \left[1/(1+\gamma)m^2c^2\right]\bar{p}(\bar{p}\cdot\bar{K}) \tag{3.8}$$

Simultaneously we obtain a relativistic expression for the internal spin vector of S at O:

$$\bar{J}' = \gamma\bar{J} - (\bar{K}\times\bar{p})/mc - \left[1/(1+\gamma)m^2c^2\right]\bar{p}(\bar{p}\cdot\bar{J}) \tag{3.9}$$

Derivation of (3.8) and (3.9) is given in the Appendix.

In order to analyze the physical meaning of the vector $\bar{J}$ more fully let us introduce the Pauli–Lubański spin 4-vector S_i:

$$mS_i = -\tfrac{1}{2}\varepsilon_{ijkl}M^{jk}P^l, \qquad \varepsilon_{ijkl} = \varepsilon_{[ijkl]}, \qquad \varepsilon_{0123} = -1 \tag{3.10}$$

Splitting again into space and time components we get

$$
\begin{aligned}
S_O &= (\bar{p}\cdot\bar{J})/mc \\
\bar{S} &= E\bar{J}/mc^2 - (\bar{K}\times\bar{p})/mc = \gamma\bar{J} - (\bar{K}\times\bar{p})/mc
\end{aligned}
\tag{3.11}
$$

If we introduce a vector

$$\bar{L} \overset{\text{def}}{=} (1/\gamma)(\bar{K}\times\bar{p})/mc = c(\bar{K}\times\bar{p})/E \tag{3.12}$$

then it is easy to realize that

$$\bar{L} = \bar{J} \Leftrightarrow S_i = 0 \tag{3.13}$$

Recalling nonrelativistic physics we define the total relativistic orbital angular momentum vector of the (closed) system S relative to an observer at O as

$$\bar{L}_{\text{orb}} \overset{\text{def}}{=} \bar{K}'\times\bar{p}/mc = \gamma(\bar{K}\times\bar{p})/mc + (1-\gamma^2)\bar{J} + \bar{p}(\bar{p}\cdot\bar{J})/m^2c^2 \tag{3.14}$$

The total relativistic angular momentum vector of S relative to an observer at O we define as

$$\bar{J}_{\text{tot}} \overset{\text{def}}{=} \bar{J}' + \bar{L}_{\text{orb}} = (1+\gamma-\gamma^2)\bar{J} + (\gamma-1)(\bar{K}\times\bar{p})/mc$$

$$+(\gamma-1)\bar{p}(\bar{p}\cdot\bar{J})/\gamma m^2c^2 \tag{3.15}$$

If $S_i=0$, i.e., $\bar{J}'=0$, i.e., $\bar{L}=\bar{J}$, then we have

$$\bar{J}_{\text{tot}}=\bar{J}=\bar{L}_{\text{orb}}=\bar{L} \tag{3.16}$$

Even if $S_i\neq 0$ while $v/c\ll 1$ we still get approximately

$$\bar{J}_{\text{tot}}\simeq\bar{J}, \qquad \bar{L}_{\text{orb}}\simeq\bar{L} \tag{3.17}$$

but in general $\bar{J}_{\text{tot}}\neq\bar{J}$ and $\bar{L}_{\text{orb}}\neq\bar{L}$ in case $S_i\neq 0$.

Alternatively we might take $\bar{L}$ in (3.12) as a definition of the total relativistic orbital angular momentum and identify $\bar{J}$ with the total relativistic angular momentum of S relative an observer at O. Then, instead, we would in general have that $\bar{J}\neq\bar{L}+\bar{J}'$, the equality being only approximately valid in the limit $v/c\ll 1$. Which alternative is more appropriate depends on the experimental situation. It remains to be investigated, from the above point of view, what is actually measured while performing experiments. We note that both alternative relativistic generalizations above in the limit $c\to\infty$ approach the same conventional nonrelativistic physical concepts. Summarizing our discussion so far we observe that the free-particle (closed system) S is at any point O in space described by the following time-independent quantities:

Defined directly—$\bar{p}$, linear momentum of S relative *an observer* at O.
Defined directly—E, energy of S relative an *observer at O*.
Equation (3.8)—$\bar{K}'/mc$, separation vector between S and O.
Equation (3.9)—$\bar{J}'$, internal angular momentum vector of S at O.
Alternative 1:
Equation (3.14)—$\bar{L}_{\text{orb}}$, orbital angular momentum vector of S relative *an observer* at O.
Equation (3.15)—$\bar{J}_{\text{tot}}$, angular momentum vector of S relative *an observer* at O.
Alternative 2:
Equation (3.12)—$\bar{L}$, orbital angular momentum vector of S relative *an observer* at O.

Defined directly—$\bar{J}$, angular momentum vector of S relative *an observer* at O.

The relativistic quantities introduced thus possess a very straightforward physical interpretation. The axial vector $\bar{J}$ and the polar vector $\bar{K}$ carry indirect information about the system S and we hope to have clarified their precise connection to the generally accepted physical notions.

4. THE TIME-DEPENDENT CASE

In the time-dependent case when the system S is not closed we are not able to construct any meaningful "position" space-time vector out of $\bar{p}(\tau)$, $E(\tau)$, $\bar{J}(\tau)$, and $\bar{K}(\tau)$ at O. Instead we arrive at the formula

$$ct = -\bar{K}(\tau-t)\bar{p}(\tau-t)/m^2c^2 + (\tau-t)E(\tau-t)/mc$$

$$\bar{r} = E(\tau-t)\bar{K}(\tau-t)/m^2c^3 + \bar{J}(\tau-t)\times\bar{p}(\tau-t)m^2c^2 - (\tau-t)\bar{p}(\tau-t)/m \tag{4.1}$$

which seems to be useless. The coordinates in (4.1) depend very heavily on how the quantities $\bar{K}$, $\bar{J}$, $\bar{p}$, E vary with time at O. We can, however, easily see that $\bar{K}'/mc$, $\bar{J}'$, $\bar{L}$, $\bar{L}_{\text{orb}}$, $\bar{J}_{\text{tot}}$ and of course $\bar{J}$, $\bar{p}$, E, albeit now time dependent, retain their previous physical meaning. Perhaps the concept of the, now momentary, separation vector between O and S requires an additional comment. Suppose, thus, that we at O perform an instantaneous boost $c^2\bar{p}/E(\tau-t)$, then the equations in (4.1) become

$$t' = \tau' - t' = \tau'/2 = \tau''$$

$$\bar{r}' = \bar{K}'(\tau'')/mc \tag{4.2}$$

where $\tau' - t' = \tau - t$ and $\bar{K}'(\tau'')$ is again expressed by the formula (3.8). $\bar{J}'(\tau'')$, $\bar{L}(\tau'')$, $\bar{L}_{\text{orb}}(\tau'')$ and $\bar{J}_{\text{tot}}(\tau'')$ are then also given by (3.9), (3.12), (3.14), and (3.15), respectively.

It is a remarkable fact that the first time derivative of $\bar{K}'/mc\gamma$ can not generally be identified with the physical velocity $c^2\bar{p}/E$ of S relative an observer at O but constitutes a new concept. It defines an apparent instantaneous "velocity" vector of S relative an observer at O. The modulus of this "velocity" vector may be greater than the speed of light c. This apparent "velocity" vector is, thus, in a certain sense related to the properties of space itself, which in turn is related to the very existence of the system S and its observer at O. In this context compare Einstein's notion of separability discussed by d'Espagnat 1979.

5. RELATIVISTIC QUANTUM PHYSICS[3]

Introducing relativistic quantum physics the ten numbers describing the system S at O become operators obeying the well-known rules for the Poincaré algebra:

$$[\hat{P}^i, \hat{P}^k]=0, \qquad [\hat{M}^{ij}, \hat{P}^k]=2ig^{k[j}\hat{P}^{i]}$$

$$[\hat{M}^{ij}, \hat{M}^{lk}]=2i(g^{l[j}\hat{M}^{i]k}+g^{k[i}\hat{M}^{j]l})$$

$$g_{00}=g^{00}=-g_{\alpha\alpha}=-g^{\alpha\alpha}=1, \qquad g^{ik}=0 \qquad \text{if } i\neq k \tag{5.1}$$

Commutation relations between all the physical quantities introduced earlier may now be deduced from (5.1). For example, in the free-particle case we put for operators of space-time coordinates:

$$\hat{X}^i=(1/m^2)\hat{M}^{ik}\hat{P}_k+\lambda\hat{P}^i \tag{5.2}$$

Using the operator identity $[\hat{A}\hat{B},\hat{C}]=\hat{A}[\hat{B},\hat{C}]+[\hat{A},\hat{C}]\hat{B}$ we get

$$[\hat{X}^i, \hat{P}^j]=-ig^{ij}+i\hat{P}^i\hat{P}^j/m^2 \tag{5.3}$$

or in more explicit terms with proper units:

$$[\hat{t}, \hat{E}]=-i\hbar+i\hbar\hat{E}^2/m^2c^4, \qquad [c\hat{t}, \hat{p}_\alpha]=[\hat{x}_\alpha, \hat{E}/c]=-i\hbar\hat{E}\hat{p}_\alpha/m^2c^3 \tag{5.4}$$

$$[\hat{x}_\alpha, \hat{p}_\beta]=i\hbar\delta_{\alpha\beta}+i\hbar\hat{p}_\alpha\hat{p}_\beta/m^2c^2 \tag{5.5}$$

If the system is not closed we have to use $\hat{\bar{K}}'/mc$ as a "position" operator. Once again using the operator identity above we get

$$[\hat{K}'_\alpha/mc, \hat{p}_\beta]=i\hbar\left\{\delta_{\alpha\beta}+\frac{1}{m^2c^2}\hat{p}_\alpha\hat{p}_\beta\left[1-\frac{\hat{E}}{mc^2}\left(1+\frac{\hat{E}}{mc^2}\right)^{-1}\right]\right\} \tag{5.6}$$

[3]From now on we put $\hbar=c=1$ if not explicitly stated otherwise.

It is now obvious that neither the pair $(\hat{x}_\alpha, \hat{p}_\beta)$ nor $(\hat{K}'_\alpha/mc, \hat{p}_\beta)$ can serve as a pair of canonically conjugate operators. In the limit $v/c \ll 1$, on the other hand, both pairs may be so used interchangeably. Looking for a covariant, physically meaningful (i.e., from the very start allowing for intrinsic spin) pair of canonically conjugate "particle" operators compatible with the whole of Poincaré algebra, we are inevitably led into Penrose's twistor theory (Penrose, 1975; Hughston, 1979; Hughston and Ward, 1979). In this paper we shall not, at least not explicitly, be concerned with this beautiful and self-contained theory. For an introductory review see, e.g., Bette (1979).

6. PARTICLE INTERACTIONS INDUCED BY POINCARÉ ALGEBRA

Operators of the squared mass and squared spin commute with all operators in the Poincaré algebra associated with the closed system S. They are given by

$$\hat{m}^2 = \hat{P}^i \hat{P}_i / m^2$$

$$\hat{s}^2 = \hat{S}^i \hat{S}_i = \tfrac{1}{4} \varepsilon_{ijkl} \varepsilon^{i}_{\ mnr} \hat{P}^r \hat{M}^{mn} \hat{P}^l \hat{M}^{jk} / m^2 \tag{6.1}$$

where m^2 is the eigenvalue of $\hat{P}^i \hat{P}_i$.

The operators in (6.1) are usually called Casimir operators of S. Having in mind the underlying symplectic twistor structure of S we assume that its Hamiltonian operator is always a function of the Casimir operators above:

$$\hat{H} = \hat{H}(\hat{m}^2, \hat{s}^2) \tag{6.2}$$

If we now imagine that our free particle S is in fact a composed system consisting of two interacting parts (subparticles) then we may formally write

$$\hat{P}^i = \hat{P}^i_{(1)}(\tau) + \hat{P}^i_{(2)}(\tau), \qquad \hat{M}^{ik} = \hat{M}^{ik}_{(1)}(\tau) + \hat{M}^{ik}_{(2)}(\tau) \tag{6.3}$$

where τ is an intrinsic time parameter at O.

Operators associated with different parts of the system S are supposed to commute with each other. Imitating nonrelativistic dynamics we obtain the following relativistic quantum mechanical equations of motion for operators describing the two interacting subparticles:

$$\tau_0 \frac{d\hat{P}^i_{(a)}}{d\tau} = -i\left[\hat{H}, \hat{P}^i_{(a)}\right], \qquad \tau_0 \frac{d\hat{M}^{ik}_{(a)}}{d\tau} = -i\left[\hat{H}, \hat{M}^{ik}_{(a)}\right] \qquad (a)=(1),(2) \tag{6.4}$$

where square brackets denote commutators corresponding to the classical twistor Poisson (curly) brackets (Penrose, 1968; Bette, 1979) and where τ_0 is a constant with the dimension of time. Let us choose a very simple form for the "Hamiltonian" in (6.2):

$$\hat{H} = \hat{m}^2 - \hat{s}^2 \tag{6.5}$$

Performing the commutation on the right-hand side of (6.4), neglecting terms of order $\hbar^2$, and letting operators become usual $(c-)$ numbers we get the following set of (quasi) classical dynamical equations:

$$\tau_0 \frac{dP^a_{(1)}}{d\tau} = 2S^i \varepsilon_{ijkl} P^l g^{a[k}_{(1)} P^{j]}/m$$

$$\tau_0 \frac{dM^{ab}_{(1)}}{d\tau} = \frac{4}{m^2} P_j g^{j}_{(1)}{}^{[a} P^{b]} + 2S^i \varepsilon_{ijkl} M^{jk} g^{l}_{(1)}{}^{[a} P^{b]}/m$$

$$+ 2S^i \varepsilon_{ijkl} P^l \left(g^{a[k}_{(1)} M^{j]b} + g^{b[j}_{(1)} M^{k]a} \right)/m$$

$$P^a_{(2)} = P^a - P^a_{(1)}, \qquad M^{ab}_{(2)} = M^{ab} - M^{ab}_{(1)} \tag{6.6}$$

Choosing a space coordinate system with its origin in the center of energy of the total system in such a way that

$$M^{21} = M^{13} = M^{\alpha 0} = 0, \qquad M^{32} = s, \quad \text{i.e.,} \quad S_0 = 0 \quad \text{and}$$

$$\bar{S} = (s, 0, 0), \qquad P_0 = m, \qquad \bar{p} = \bar{0} \tag{6.7}$$

the equations in (6.6) become

$$\frac{dP^0_{(1)}}{d\tau}=0, \qquad \frac{dP^1_{(1)}}{d\tau}=0, \qquad \tau_0\frac{dP^2_{(1)}}{d\tau}=-2sP^3_{(1)}, \qquad \tau_0\frac{dP^3_{(1)}}{d\tau}=2sP^2_{(1)} \tag{6.8}$$

$$\tau_0\frac{dM^{10}_{(1)}}{d\tau}=-\frac{2}{m}P^1_{(1)}, \qquad \tau_0\frac{dM^{20}_{(1)}}{d\tau}=2\frac{(-1-s^2)}{m}P^2_{(1)}+2sM^{30}_{(1)} \tag{6.9}$$

$$\tau_0\frac{dM^{30}_{(1)}}{d\tau}=2\frac{(-1-s^2)}{m}P^3_{(1)}-2sM^{20}_{(1)}$$

$$\frac{dM^{\alpha\beta}_{(1)}}{d\tau}=0 \tag{6.10}$$

Putting

$$\omega_0=2s/\tau_0, \qquad -P^\alpha_{(1)}=p_\alpha=\mu\gamma v_\alpha, \qquad J_\sigma=\tfrac{1}{2}\varepsilon_{\alpha\beta\sigma}M^{\alpha\beta}_{(1)}$$

$$K_\alpha=M^{\alpha 0}_{(1)}, \qquad P^0_{(1)}=E=\mu\gamma \tag{6.11}$$

where μ is the rest mass of particle No. (1) we easily get the solutions of (6.8)–(6.10):

$$p_2=A\cos\omega_0\tau-B\sin\omega_0\tau$$

$$p_3=A\sin\omega_0\tau+B\cos\omega_0\tau$$

$$K_2=\left(A\frac{1+s^2}{ms}+D\right)\sin\omega_0\tau+\left(B\frac{1+s^2}{ms}-C\right)\cos\omega_0\tau$$

$$K_3=C\sin\omega_0\tau+D\cos\omega_0\tau$$

$$p_1=\text{const}$$

$$K_1=\frac{2}{m\tau_0}p_1\tau+F$$

$$E=\text{const}$$

$$\bar{J}=\text{const} \tag{6.12}$$

From the above solutions we can form all (relativistic) physical quantities introduced earlier in Section 3. Their exact temporal evolution depends on the numerical relations between the integration constants A, B, C, D, F, μ, $|\bar{p}|$, and $\bar{J}$.

If, on the other hand, $s \gg 1$, $v/c \ll 1$, and $\tau_0 = 2(1+s^2)/m$ we get approximately

$$\dot{K}_2/\mu \simeq v_2, \qquad \dot{K}_3/\mu \simeq v_3, \qquad \dot{K}_1 \simeq 0 \tag{6.13}$$

where the dot denotes the time derivative.

From this we realize that in the nonrelativistic, large angular momentum limit, the two velocity concepts mentioned at the end of Section 3 coincide.

In the approximation above [equations (6.8) and (6.13)] we also note that, while one of the (sub)particles performs a spiral motion inwards, the other is spiralling outwards in the opposite direction. We suggest the term "twist interaction" for the kind of interaction described here and argue that for small values of angular momentum (and also for high values of the total energy), i.e., in connection with elementary particles, the approach above is more appropriate than the conventional one.

Assuming different relative numerical ratios between integration constants in the solutions one might easily explore their exact physical meaning. This we intend to do in a forthcoming paper.

Conclusions and remarks: The proper treatment of the interaction presented here requires the use of the quantum theory of twistors. Penrose and co-workers are intensely involved in the construction of an appropriate mathematical apparatus by which such a treatment can be made possible. Many questions are, however, still unsolved.

While awaiting the new mathematical apparatus of Penrose's school to become sufficiently developed, we may in some cases tentatively proceed as suggested in this paper. For example, we could treat a spinning (note that the nonvanishing of the total spin of the system is crucial for the interaction to exist) free particle as one composed of two interacting twistors where interaction is generated by the procedure analogous to that presented here. The behavior of elementary particles could then perhaps be explained, at least qualitatively, by means of the (quasiclassical) solutions obtained in this manner.

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APPENDIX

A 4-vector A_i consists of 3-vector $\bar{A}=(A_1, A_2, A_3)$ and an associated scalar A_0. Under boost transformations the scalar A_0 and the 3-vector $\bar{A}$ mix with each other according to the rule (Rindler, 1960)

$$\bar{A}'=\bar{A}+\bar{v}\left\{\frac{\bar{v}\cdot\bar{A}}{v^2}(\gamma-1)-\frac{\gamma}{c}A_0\right\}$$

$$A'_0=\gamma\left(A_0-\frac{\bar{v}\cdot\bar{A}}{c}\right) \tag{A.1}$$

From (A.1) we easily deduce the transformation law for an antisymmetric tensor M_{ik} of second rank. M_{ik} consists of an axial 3-vector $\bar{J}$ and a polar 3-vector $\bar{K}$. Forming M_{ik} by taking two different arbitrary 4-vectors $M_{ik}=2A_{[i}B_{k]}$ and using (A.1) we get for the polar vector part $\bar{K}=(M_{01}, M_{02}, M_{03})$

$$M'_{0\alpha}=\gamma\left\{M_{0\alpha}+(1/c)v_\beta M_{\alpha\beta}-\left[\gamma/(1+\gamma)c^2\right]v_\alpha v_\beta M_{0\beta}\right\} \tag{A.2}$$

or equivalently

$$\bar{K}'=\gamma\left\{\bar{K}+(\bar{J}\times\bar{v})/c-\left[\gamma/(1+\gamma)c^2\right]\bar{v}(\bar{v}\cdot\bar{K})\right\} \tag{A.3}$$

Similarly for the axial vector part $\bar{J}=(M_{32}, M_{13}, M_{21})$ we obtain

$$\bar{J}'=\gamma\left\{\bar{J}-(\bar{K}\times\bar{v})/c-\left[\gamma/(1+\gamma)c^2\right]\bar{v}(\bar{v}\cdot\bar{J})\right\} \tag{A.4}$$

Note also that (A.3) and (A.4) automatically provide us with the exact relativistic transformation formulas (in the 3-vector form) for the electric $\bar{E}$ and the magnetic $\bar{H}$ 3-vectors, respectively.

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